Quantum Inequalities and Particle Creation

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General Relativity

- The spacetime is treated as a classical, curved, Lorentzian manifold which is governed by the Einstein equation,

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} - \Lambda g_{\mu\nu}}_{\text{system of } 2^{\text{nd}} \text{ order PDE's}} = 8\pi G_N \underbrace{T_{\mu\nu}}_{\text{source}} .$$

- Let u^μ be any future-directed timelike vector, and k^μ be any future-directed null vector. Then, the stress-tensor for classical matter is postulated to obey a group of classical energy conditions:
  - W.E.C. $T_{\mu\nu} u^\mu u^\nu \geq 0$
  - N.E.C. $T_{\mu\nu} k^\mu k^\nu \geq 0$
  - S.E.C. $(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) u^\mu u^\nu \geq 0$
  - D.E.C. $T^\mu{}_\nu u^\nu$ is a timelike or null vector.

General Relativity

- The classical energy condition were the assumptions for the proofs of the *singularity theorems* of Penrose and Hawking
- One can still prove the singularity theorems under "weaker" averaged energy conditions.
- Let $\gamma^\mu(\tau)$ be a future-directed timelike geodesic parameterized by proper time τ, then the four-velocity tangent to the geodesic is $u^\mu(\tau) = d\gamma^\mu(\tau)/d\tau$, and the average weak energy condition is
  - A.W.E.C. $$\int_\gamma [T_{\mu\nu} \circ \gamma(\tau)]\, u^\mu(\tau)\, u^\nu(\tau)\, d\tau \geq 0$$
- There is also an average null energy condition;
  - A.N.E.C. $$\int_\gamma [T_{\mu\nu} \circ \gamma(\lambda)]\, k^\mu(\lambda)\, k^\nu(\lambda)\, d\lambda \geq 0$$

Quantum Field Theory in Curved Spacetime

- The spacetime is still treated as a classical, curved, Lorentzian manifold.
- We study the behavior of relativistic quantum field theories propagating on this background spacetime. (Klein-Gordon, E&M, Proca, Dirac, spin-2, p-forms,...)
- To first order, we are neglecting the back reaction of the quantum field on the curvature of the spacetime.
- We are then interested in things like particle creation and the expectation value of the stress-tensor operator for a given quantum state $|\omega\rangle$, i.e.,

$$\langle\omega|\, \boldsymbol{T}_{\mu\nu} |\omega\rangle_{\mathrm{Ren.}}.$$

- It is a general feature of QFT that EVERY classical energy condition can be violated[1], even the averaged ones. (Casimir vacuum state, squeezed states, ...)

1. H. Epstein, V. Glaser, and A. Jaffe, Il Nuovo Cim. **36**, 1016 (1965).

Quantum Inequalities

- Quantum inequalities are one "natural" replacement for the classical energy conditions.
- They were first proposed by Ford[2], and then proven by Ford and Roman[3] in 1995.
- They found, for the two-dimensional cylinder spacetime,

$$\frac{\tau_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle : \boldsymbol{T}_{\mu\nu}u^{\mu}u^{\nu} : \rangle_{\omega}}{\tau^2+\tau_0^2}d\tau \geq -\frac{1}{8\pi\tau_0^2}.$$

- Various forms of quantum inequalities have been extensively studied over the last two decades. They are derived directly from QFT without recourse to the standard uncertainty relationships.

2. L. H. Ford, Proc. Roy. Soc. Lond. A **364**, 227 (1978).
3. L. H. Ford and T. A. Roman, Phys. Rev. D **51**, 4277 (1995).

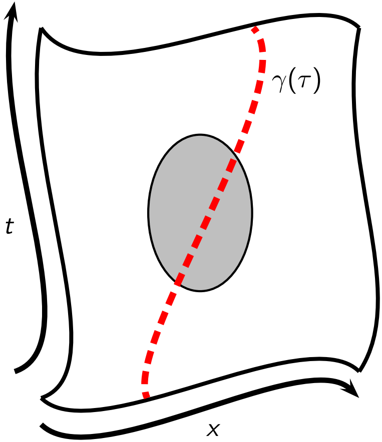


Figure: A universe with a timelike geodesic (red line) passing through a region of space containing negative energy density (gray region).

Normal-ordered energy density

$$\langle :\boldsymbol{\rho}:\rangle_\omega(\tau) = \langle :\boldsymbol{T}_{\alpha\beta}u^\alpha u^\beta :\rangle_\omega(\tau)$$

Sampling Functions

$$g(\tau) \in C_0^\infty(\mathbb{R})$$

Worldline Quantum Inequality

$$\int_I \langle :\boldsymbol{\rho}:\rangle_\omega\, g^2(\tau)\, d\tau \geq -Q_{\omega_0}(g)$$

The right hand side of the inequality is finite.

Worldline Quantum Inequalities

The most studied form of quantum energy inequality is for averaging along the worldline of an inertial observer:

- Massless Scalar Field in 2-Dimensional Minkowski Spacetime:[4]

$$\int_I \langle : \rho(\tau) : \rangle_\omega \, g(\tau)^2 d\tau \geq -\frac{1}{4\pi} \int_I [g'(\tau)]^2 d\tau$$

- Electromagnetic Field in 4-Dimensional Minkowski Spacetime:[5]

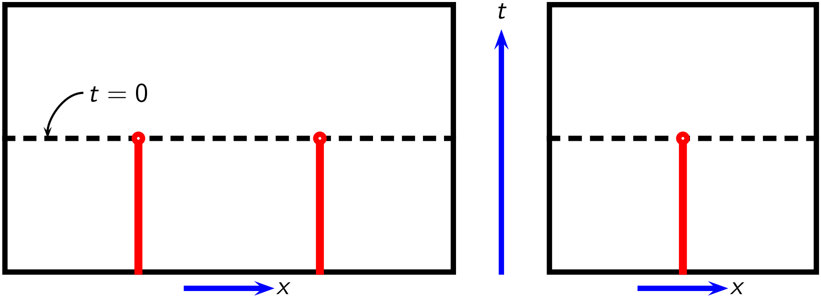
$$\int_I \langle : \rho(\tau) : \rangle_\omega \, g(\tau)^2 d\tau \geq -\frac{1}{8\pi^2} \int_I [g''(\tau)]^2 d\tau$$

- Also proven for various fields in curved spacetimes. (Scalar, Electromagnetic, Dirac, p-form, Spin-2)

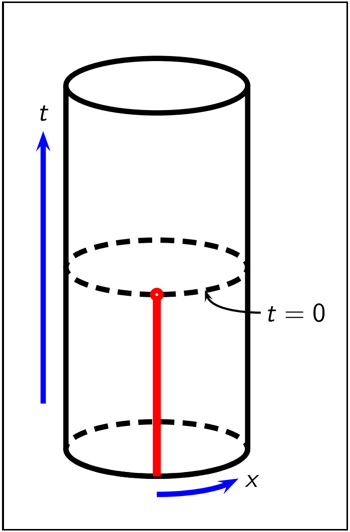
4. C. J. Fewster & S. P. Eveson, Phys. Rev. D **58**, 084010 (1998).
5. M. J. Pfenning, Phys. Rev. D **65**, 024009 (2002).

Motivation

- In the winter of 2011, Dan Solomon[4] published a paper in which he claims to have a model that violates the worldline QI in two dimensions.
- Solomon followed this with a second paper[5] where he claims to have another model that violates the spatial QI.



4. D. Solomon, Adv. Stud. Theor. Phys. **5**, no. 5, 227-251 (2011). 5. D. Solomon, Adv. Stud. Theor. Phys. **6**, no. 6, 245-262 (2012).



Model Spacetime

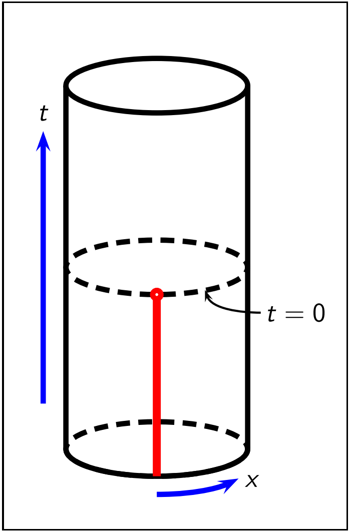
- $M \simeq \mathbb{R} \times S^1$
  Circumference of universe is L
- Scalar Field with Potential

  $$\left[\partial_t^2 - \partial_x^2 + V(x,t)\right] \Phi(x,t) = 0$$

- Potential

  $$V(x,t) = 2\xi_0 \delta(x) \Theta(-t)$$

  $\xi_0 > 0$ is the coupling constant



OUT Region

$$\left[\partial_t^2 - \partial_x^2\right] \Phi^{\mathrm{OUT}}(x,t) = 0$$

Matching Solutions

$$\Phi^{\mathrm{IN}}(x,0) = \Phi^{\mathrm{OUT}}(x,0)$$

$$\partial_t \Phi^{\mathrm{IN}}(x,0) = \partial_t \Phi^{\mathrm{OUT}}(x,0)$$

IN Region

$$\left[\partial_t^2 - \partial_x^2 + 2\xi_0 \delta(x)\right] \Phi^{\mathrm{IN}}(x,t) = 0$$

IN Region Eigenfunctions

Odd Modes (positive frequency)

$$\phi^{\mathrm{odd}}(n,x,t) = (k_n L)^{-1/2} \sin(k_n x)\, e^{-ik_n t}$$

$$k_n = \frac{2\pi n}{L} \qquad \text{where} \qquad n = 1, 2, 3, \ldots$$

Even Modes (positive frequency)

$$\phi^{\mathrm{even}}(j,x,t) = (\kappa_j L)^{-1/2} A_j \left[\cos(\kappa_j x) + \frac{\xi_0}{\kappa_j}\sin(\kappa_j |x|)\right] e^{-i\kappa_j t}$$

$$\kappa_j = \frac{2Z_j}{L} \qquad \text{where} \qquad j = 1, 2, 3, \ldots$$

A_j is a normalization constant, Z_j is the j-th root of a transcendental eq.

Negative frequency modes are given by the complex conjugate.

IN Region Eigenvalues

$$\frac{1}{\chi} Z = \cot(Z) \qquad \text{where} \qquad \chi = \frac{\xi_0 L}{2}$$

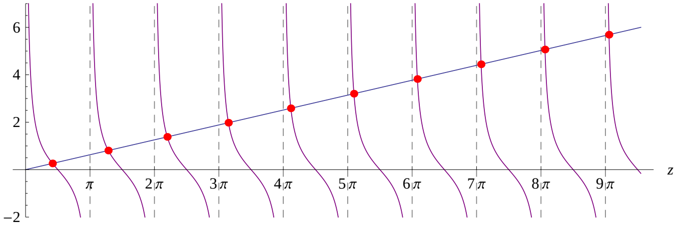


Figure: Graphical determination of the even eigenvalues from the transcendental equation. The values $L = 1$ and $\xi_0 = 10$ were used. $(\chi = 5)$

Asymptotically,for very large j we find $Z_j \simeq \pi(j-1) + \dfrac{\chi}{\pi(j-1)}$.

OUT Region Eigenfunctions

Odd Modes (positive frequency)

$$\psi^{\rm odd}(n,x,t) = (k_n L)^{-1/2} \sin(k_n x)\, e^{-ik_n t}$$

Even Modes (positive frequency)

$$\psi^{\rm even}(n,x,t) = (k_n L)^{-1/2} \cos(k_n x)\, e^{-ik_n t}$$

$$k_n = \frac{2\pi n}{L} \qquad \text{where} \qquad n = 1, 2, 3, \ldots$$

Topological Mode (zero frequency)

$$\psi^{\rm top.}(x,t) = \sqrt{\frac{\ell}{2L}} \left(1 - i\frac{t}{\ell}\right)$$

Negative frequency modes are given by the complex conjugate.

Mode Functions for the Entire Spacetime

Odd Modes

$$\Phi^{\mathrm{odd}}(n,x,t) = (k_n L)^{-1/2} \sin(k_n x)\, e^{-ik_n t}$$

Even Modes

$$\Phi^{\mathrm{even}}(j,x,t) = \begin{cases} \phi^{\mathrm{even}}(j,x,t) & \text{for } t \leq 0, \\ \phi^{\mathrm{even}}_{OUT}(j,x,t) & \text{for } t > 0. \end{cases}$$

$$\begin{aligned} \phi^{\mathrm{even}}_{\mathrm{OUT}}(j,x,t) &= \overline{\alpha_{0j}}\,\psi^{\mathrm{top.}}(x,t) - \beta_{0j}\,\overline{\psi^{\mathrm{top.}}(x,t)} \\ &\quad + \sum_{n=1}^{\infty}\left[\overline{\alpha_{nj}}\,\psi^{\mathrm{even}}(n,x,t) - \beta_{nj}\,\overline{\psi^{\mathrm{even}}(n,x,t)}\right] \end{aligned}$$

$\overline{\alpha_{0j}}$, β_{0j}, $\overline{\alpha_{nj}}$, and β_{nj} are Bogolubov coefficients

Canonical Second Quantization Method

Classical Physics of Model

- In the Hamiltonian formulation, the observables are the real–valued field $\Phi(x,t)$ and the canonically conjugate momenta $\partial_t \Phi(x,t)$.

$$\begin{aligned} \Phi(x,t) &= \sum_{n=1}^{\infty}\left[a_n \Phi^{\mathrm{odd}}(n,x,t) + \overline{a_n \Phi^{\mathrm{odd}}(n,x,t)}\right] \\ &\quad + \sum_{j=1}^{\infty}\left[b_j \Phi^{\mathrm{even}}(j,x,t) + \overline{b_j \Phi^{\mathrm{even}}(j,x,t)}\right] \end{aligned}$$

- In addition, define a fully anytisymmetric bilinear form

$$\sigma(\Phi_1,\Phi_2) \equiv \int_{\mathrm{S}^1} i^*(\Phi_1 \partial_t \Phi_2 - \Phi_2 \partial_t \Phi_1)$$

- Together, these two things yield a suitable symplectic phase space.

Canonical Second Quantization Method

Second Quantization of Model

- Promote the field to a self-adjoint operator.

$$\begin{aligned}\mathbf{\Phi}(x,t) &= \sum_{n=1}^{\infty}\left[\boldsymbol{a}_n \Phi^{\text{odd}}(n,x,t) + \boldsymbol{a}_n^{\dagger}\overline{\Phi^{\text{odd}}(n,x,t)}\right] \\ &\quad + \sum_{j=1}^{\infty}\left[\boldsymbol{b}_j \Phi^{\text{even}}(j,x,t) + \boldsymbol{b}_j^{\dagger}\overline{\Phi^{\text{even}}(j,x,t)}\right]\end{aligned}$$

- $\boldsymbol{a}_n^{\dagger}$ and $\boldsymbol{b}_j^{\dagger}$ are the operators which create particles. $\boldsymbol{a}_n$ and $\boldsymbol{b}_j$ are the operators which annihilate particles.
- Commutator Relations

$$[\boldsymbol{a}_n, \boldsymbol{a}_m^{\dagger}] = \delta_{nm}\mathbb{I} \qquad \text{and} \qquad [\boldsymbol{b}_j, \boldsymbol{b}_{j'}^{\dagger}] = \delta_{jj'}\mathbb{I}$$

Canonical Second Quantization Method

State Space

- $|0\rangle$ is the *IN* vacuum state. (No particles present)
- $|1_n\rangle$ one particle in the antisymmetric mode with eigenvalue n.
- $|2_j\rangle$ two particles in the symmetric mode with eigenvalue j.
- $|1_n, 1_j\rangle$ two particles present in different modes.

Creation Operator

$$\boldsymbol{a}_n^\dagger|0\rangle = |1_n\rangle$$

$$\boldsymbol{a}_n^\dagger|1_n\rangle = \sqrt{2}|2_n\rangle$$

$$\boldsymbol{a}_n^\dagger|2_j\rangle = |1_n, 2_j\rangle$$

$$\boldsymbol{a}_n^\dagger|1_n, 1_j\rangle = \sqrt{2}|2_n, 1_j\rangle$$

Annihilation Operator

$$\boldsymbol{a}_n|0\rangle = 0$$

$$\boldsymbol{a}_n|1_n\rangle = |0\rangle$$

$$\boldsymbol{a}_n|2_j\rangle = 0$$

$$\boldsymbol{a}_n|1_n, 1_j\rangle = |1_j\rangle$$

Alternative Representation for *OUT* Region

Second Representation for the Field Operator

$$\mathbf{\Psi}(x,t) = \widetilde{\boldsymbol{a}}\,\psi^{\text{top.}}(x,t) + \widetilde{\boldsymbol{a}}^{\dagger}\,\overline{\psi^{\text{top.}}(x,t)} + \sum_{n=1}^{\infty}\Big[\widetilde{\boldsymbol{a}}_n\,\psi^{\text{odd}}(n,x,t) + \widetilde{\boldsymbol{a}}_n^{\dagger}\,\overline{\psi^{\text{odd}}(n,x,t)} + \widetilde{\boldsymbol{b}}_n\,\psi^{\text{even}}(n,x,t) + \widetilde{\boldsymbol{b}}_n^{\dagger}\,\overline{\psi^{\text{even}}(n,x,t)}\Big].$$

Second Set of Operators

$\tilde{\boldsymbol{a}}^{\dagger}$	and	$\tilde{\boldsymbol{a}}$
$\tilde{\boldsymbol{a}}_n^{\dagger} = \boldsymbol{a}_n^{\dagger}$	and	$\tilde{\boldsymbol{a}}_n = \boldsymbol{a}_n$
$\tilde{\boldsymbol{b}}_n^{\dagger} \neq \boldsymbol{b}_j^{\dagger}$	and	$\tilde{\boldsymbol{b}}_n \neq \boldsymbol{b}_j$

Second Set of States

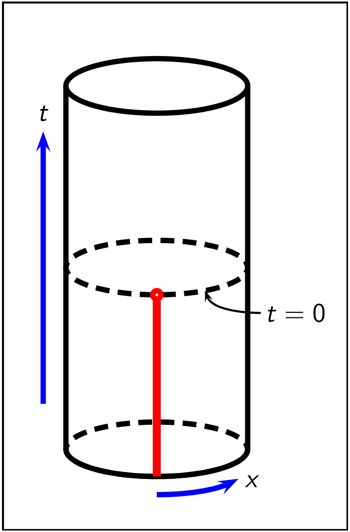
$|\tilde{0}\rangle$ is the *OUT* vacuum state.

$|\tilde{1}_n\rangle$ is a one particle state.

$|\tilde{2}_n\rangle$ is a two particle state.

$|\tilde{1}_n, \tilde{1}_{n'}\rangle$ is a two particle state.

Quantum Model



OUT Region ($t > 0$)

Field Operators: $\mathbf{\Phi}(x,t)$ and $\mathbf{\Psi}(x,t)$
Vacuum States: $|0_L\rangle$ and $|\tilde{0}_L\rangle$
Creation and annihilation operators $\times 2$.

Matching Solutions ($t = 0$)

The operators and states between the two representations are linked by the Bogolubov coefficients

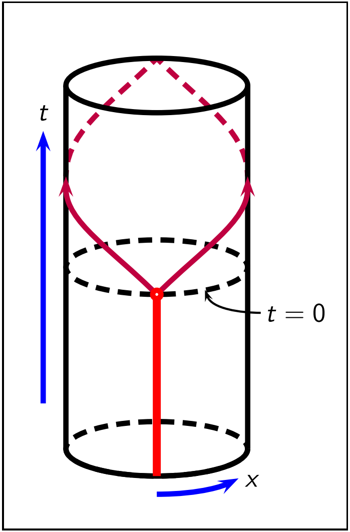
IN Region ($t < 0$)

Field Operator: $\mathbf{\Phi}(x,t)$
IN Vacuum State: $|0_L\rangle$
Creation and annihilation operators.

Particle Creation

- Before $t = 0$, let the particle state of the universe be the *IN* vacuum state $|0_L\rangle$.
- In the Heisenberg picture, the operators evolve in time while the states are time-independent, therefore, the state in the *OUT* region remains $|0_L\rangle$ forever.
- However, the "TRUE" vacuum in the *OUT* region is $|\tilde{0}_L\rangle$. The "TRUE" particle states are the ones with respect to the tilded-operators.
- Since the *IN* vacuum state $|0_L\rangle$ can be re-expressed as a linear superposition of the "TRUE" particles for the *OUT* region, we find that for times $t > 0$ there is non-zero probability of finding "TRUE" even-mode particles in the *OUT* region.
- This is interpreted as **PARTICLE CREATION** due to the shutting off of the potential.
- No "TRUE" odd-mode particles are created in the shut-off.

Recap of Quantum Model



OUT Region ($t > 0$)

Particle State: $|0_L\rangle$
OUT Vacuum State: $|\tilde{0}_L\rangle$
Particles are present.

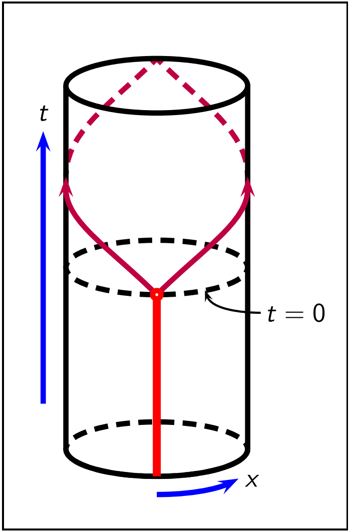
Potential Shutoff ($t = 0$)

The energy for the production of the particles comes out of the vacuum.

IN Region ($t < 0$)

Particle State: $|0_L\rangle$
IN Vacuum State: $|0_L\rangle$
No particles are present.

Casimir Effect



OUT Region ($t > 0$)

$$\langle \widetilde{0}_L | \boldsymbol{T}_{\mu\nu} | \widetilde{0}_L \rangle_{\text{Ren.}} = \left(-\frac{\pi}{6L^2} + \frac{1}{4\ell L} \right) \delta_{\mu\nu}$$

Expression for a potential free cylinder spacetime.

IN Region ($t < 0$)

$$\langle 0_L | \boldsymbol{T}_{\mu\nu} | 0_L \rangle_{Ren.} = \left(-\frac{\pi}{6L^2} + \frac{\mathcal{A}}{L^2} \right) \delta_{\mu\nu},$$

- This holds everywhere except at the location of the delta-function potential.
- $\mathcal{A} > 0$ and is a function of χ.

Renormalized Stress-Tensor ($t > 0$)

After an enormous amount of algebra, one finds that the renormalized expectation value of the stress-tensor operator for the *IN* vacuum state on the *OUT* region is

$$\begin{aligned}\langle 0_L | \boldsymbol{T}_{\mu\nu} | 0_L \rangle_{Ren.} &= \left(-\frac{\pi}{6L^2} + \frac{\mathcal{B}-\mathcal{C}}{L^2} \right) \delta_{\mu\nu} \\ &+ \frac{\mathcal{C}}{2L^2} \sum_{n=-\infty}^{\infty} \left[\delta\left(\frac{t+x}{L} - n \right) + \delta\left(\frac{t-x}{L} - n \right) \right] \delta_{\mu\nu} \\ &+ \frac{\mathcal{C}}{2L^2} \sum_{n=-\infty}^{\infty} \left[\delta\left(\frac{t+x}{L} - n \right) - \delta\left(\frac{t-x}{L} - n \right) \right] \sigma_{\mu\nu},\end{aligned}$$

where

$$\mathcal{B} = \mathcal{B}(\chi) \geq 0, \qquad \mathcal{C} = \frac{\chi}{\pi}, \qquad \text{and} \qquad \sigma_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

All of the expressions above are independent of ℓ!

Classical Energy Conditions

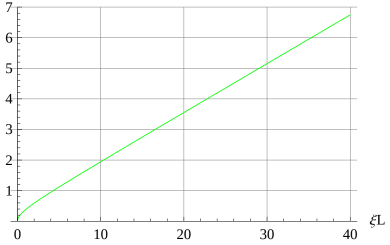
- All the classical energy conditions for this model can fail. For example, WEC

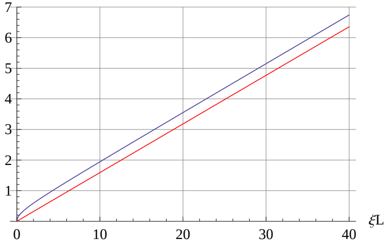
$$\langle 0_L | \boldsymbol{T}_{\mu\nu} | 0_L \rangle_{Ren.} u^\mu u^\nu \ngeq 0$$

whenever

$$\mathcal{B}(\chi) - \mathcal{C}(\chi) < \frac{\pi}{6}.$$

- Actually, all the classical energy conditions fail simultaneously under the above condition.





Quantum Inequality

Recall, the QI is a constraint on the difference of expectation values for two separate states. Therefore, we choose to look at the difference for the states $|0_L\rangle$ and $|\tilde{0}_L\rangle$. Define

$$\langle \Delta\rho \rangle_{0_L}(\tau) \equiv \left[\langle 0_L | \boldsymbol{T}_{\mu\nu} | 0_L \rangle - \langle \tilde{0}_L | \boldsymbol{T}_{\mu\nu} | \tilde{0}_L \rangle \right] u^\mu u^\nu$$

then

The worldline QI for $\mathbb{R} \times S^1$

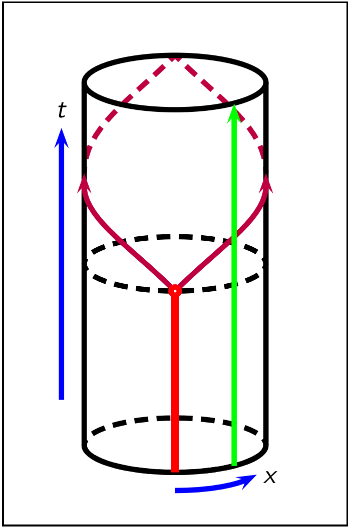
$$\int_{\mathbb{R}} \langle \Delta\rho \rangle_{0_L} [g(\tau)]^2 \, d\tau \geq \frac{1+v^2}{1-v^2} \left(-\frac{1}{4\ell L} \right) \int_{\mathbb{R}} [g(\tau)]^2 \, d\tau - \mathbb{Q}(g)$$

The left hand side evaluates to

$$L.H.S. = \frac{1+v^2}{1-v^2} \left(-\frac{1}{4\ell L} \right) \int_{\mathbb{R}} [g(\tau)]^2 \, d\tau + (\text{positive terms})$$

The QI is satisfied for all inertial observers!

Outstanding Issues

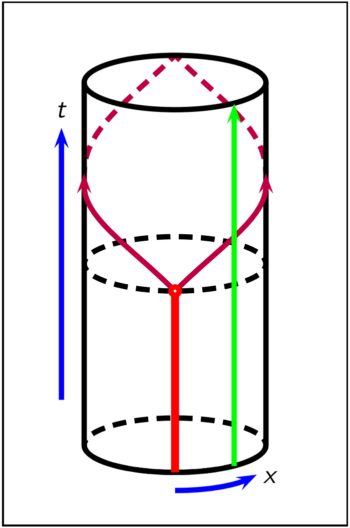


Remaining to do...

analytically prove or disprove the behaviors seen in the numerical simulations.

- Does $\mathcal{A} = \mathcal{B}$?
- Is $\mathcal{B} - \mathcal{C} \leq -\frac{\pi}{6}$ for all values of χ?

Conclusions



So far, I have...

- determined a complete basis of eigenfunctions for the *IN* and *OUT* regions which can be quantized,
- second quantized the model and determined the particle creation at the moment that the potential collapses,
- calculated the renormalized stress-tensor and shown it can violate all the classical energy conditions
- and proven that the stress-tensor obeys the worldline QI, contrary to what Solomon claims.

Thank You.

Questions?

Mathematical Curiosity?

In several places in my work, I kept coming across series of the form

$$F_p(\chi) = \chi^2 \sum_{j=1}^{\infty} \frac{A_j^2}{Z_j^p}$$

where the range of χ is in $[0, \infty)$, the power $p > 1$, the

$$A_j^2 = \frac{Z_j^2}{Z_j^2 + \chi^2 + \chi}, \qquad \text{and} \qquad Z_j = \chi \cot(Z_j).$$

The F_p's satisfy a recurrence-like formula

$$(\chi^2 + \chi) F'_{p+2}(\chi) + (p-1) F_{p+2}(\chi) + F'_p(\chi) - \frac{2}{\chi} F_p(\chi) = 0.$$

Interestingly, there are analytic expressions for p an even integer

$$F_2(\chi) = \frac{\chi}{2}, \qquad F_4(\chi) = \frac{1}{2}, \qquad F_6(\chi) = \frac{1}{2}\left(\frac{1}{\chi} + \frac{1}{3}\right), \qquad \ldots$$

IN Region Eigenfunctions (graphical)

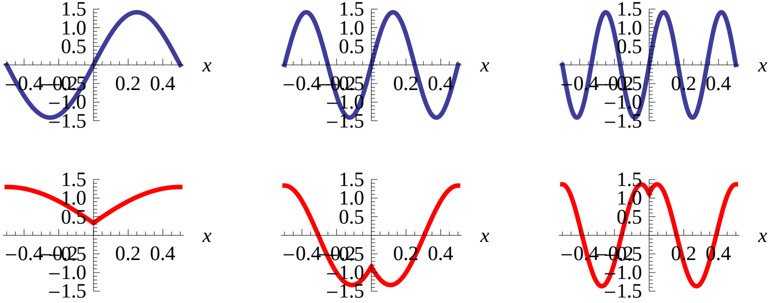


Figure: Snapshots of the first three odd eigenfunctions (top row in blue) and the first three even eigenfunctions (bottom row in red). The values $L = 1$ and $\xi_0 = 10$ were used.

Bogolubov Coefficients

$$A_j \equiv \cos(Z_j)\left[1 + \frac{\sin(Z_j)\cos(Z_j)}{Z_j}\right]^{-1/2}$$

$$Y_{j,0} \equiv \frac{\sqrt{2}\chi A_j}{Z_j^2} \qquad Y_{j,n} \equiv \frac{2\chi A_j}{Z_j^2 - (\pi n)^2}$$

$$\sum_{j=1}^{\infty} Y_{j,m} Y_{j,n} = \delta_{mn}$$

$$\overline{\alpha_{0j}} = \frac{1}{2\sqrt{\kappa_j \ell}}(\kappa_j \ell + 1) Y_{j,0} \qquad \beta_{0j} = \frac{1}{2\sqrt{\kappa_j \ell}}(\kappa_j \ell - 1) Y_{j,0}$$

$$\overline{\alpha_{nj}} = \frac{1}{2}\sqrt{\frac{k_n}{\kappa_j}}\left(\frac{\kappa_j}{k_n} + 1\right) Y_{j,n} \qquad \beta_{nj} = \frac{1}{2}\sqrt{\frac{k_n}{\kappa_j}}\left(\frac{\kappa_j}{k_n} - 1\right) Y_{j,n}$$

Number of Even-Mode Particles Created

	$\langle 0\vert\widetilde{\boldsymbol{N}}_n\vert 0\rangle = \sum_{j=1}^{\infty} \vert\beta_{nj}\vert^2$			
n	$\xi_0 = 1$	$\xi_0 = 5$	$\xi_0 = 10$	$\xi_0 = 100$
0	0.023987	0.255469	0.416834	1.082297
1	0.003875	0.024742	0.047086	0.198755
2	0.000665	0.005465	0.011781	0.070152
3	0.000231	0.002154	0.004975	0.036841
4	0.000108	0.001091	0.002639	0.022904
5	0.000059	0.000637	0.001594	0.015659
6	0.000036	0.000408	0.001048	0.011386

Table: Values generated using $L = 1$ and summing the first 500 terms in the series using Mathematica. The $n = 0$ values are determined with $\ell = L$.